Received: *September 1, 2000,* Accepted: *October 4, 2000*

On the short-distance structure of irrational non-commutative gauge theories

Shmuel Elitzur and Eliezer Rabinovici

Racah Institute of Physics, The Hebrew University
Jerusalem 91904, Israel
E-mail: elitzur@vms.huji.ac.il, eliezer@vms.huji.ac.il

Boris Pioline*

Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA
E-mail: pioline@physics.harvard.edu

Abstract: As shown by Hashimoto and Itzhaki in hep-th/9911057, the perturbative degrees of freedom of a non-commutative Yang-Mills theory (NCYM) on a torus are quasi-local only in a finite energy range. Outside that range one may resort to a Morita equivalent (or T-dual) description appropriate for that energy. In this note, we study NCYM on a non-commutative torus with an irrational deformation parameter θ. In that case, an infinite tower of dual descriptions is generically needed in order to describe the UV regime. We construct a hierarchy of dual descriptions in terms of the continued fraction approximations of θ. We encounter different descriptions depending on the level of the irrationality of θ and the amount of non-locality tolerated. The behavior turns out to be isomorphic to that found for the phase structure of the four-dimensional Villain $\mathbb{Z}_N$ lattice gauge theories, which we revisit as a warm-up. At large 't Hooft coupling, using the AdS/CFT correspondance, we find that there are domains of the radial coordinate U where no T-dual description makes the derivative expansion converge. The radial direction obtains multifractal characteristics near the boundary of AdS.

Keywords: Lattice Quantum Field Theory, Duality in Gauge Field Theories, AdS/CFT Correspondance.

*On leave of absence from LPTHE, Université Pierre et Marie Curie, PARIS VI and Université Denis Diderot, PARIS VII, Boîte 126, Tour 16, 1^{er} étage, 4 place Jussieu, F-75252 Paris CEDEX 05, France

Contents

1. Introduction

The non-commutativity introduced by the Moyal-Weyl star product appears to give a consistent deformation of supersymmetric gauge theories [1, 2, 3], and to capture part of the non-locality of open strings at least when they propagate in non-trivial gauge backgrounds [2, 4, 5, 6]. Whether it has any connection with the non-locality of quantum gravity is unclear at this stage. It nevertheless offers an interesting opportunity to study issues of non-locality in field theories.

A simple non-commutative manifold is the fuzzy torus, with a uniform non-commutativity scale Δ. Particles then behave as wave packets of Compton wavelength $1/E$ along their direction of motion, but $E\Delta^2$ in the orthogonal direction. They can be interpreted as electric dipoles transverse to their motion [7]. For a rational deformation parameter $\theta = \Delta^2/\Sigma^2$, where Σ is the size of the box, non-commutativity does not imply non-locality at short distances however: it can be eliminated at the expense of enlarging the gauge group and introducing a magnetic flux. This classical equivalence, known as Morita's [8, 9, 10], is assumed to hold true quantum mechanically for maximally supersymmetric gauge theories, and descends directly from the T-duality of open strings [9]. This is not to say that the Moyal deformed gauge theory description is altogether redundant: instead, as explained in ref. [11], it may serve in giving an appropriate quasilocal description of the degrees of freedom in a finite energy range $1/\Sigma < E < 1/(\theta\Sigma)$, in which the Compton wave length and its dual are smaller than the size of the box. Beyond that range, one may find a Morita equivalent description, where the elementary particles are again localized on the scale of the box. For θ rational, the dual $(3+1)$-dimensional commutative Yang-Mills theory then controls the ultraviolet behavior, while in the infrared the gauge theory undergoes a dimensional reduction to $(1+1)$-dimensional commutative Yang-Mills. The intermediate energy range can be covered by a finite number of Morita dual patches (a single one in the simplest rational case of $\theta = 1/s$) [11].

From the discussion above, the only situation where the non-locality of the Moyal product may be effective is clearly when the deformation parameter θ is irrational: the study of this case is the primary goal of this note. Before discussing the case of non-commutative Yang-Mills theories with irrational θ, a word is probably needed on the relevance of such considerations for physics. Indeed, for all practical purposes, any irrational number can always be approached by a rational number, and phase transitions are not expected to occur in a dense subspace of parameter space. Moreover, no way has yet been devised to actually establish a measurable parameter as irrational. Nevertheless, the difference between rational and irrational can sometimes be hinted at by going to sufficiently high energy. The case of Villain $\mathbb{Z}_N$ lattice gauge theories [12] in the presence of a θ angle (not to be confused with the non-commutativity parameter) is particularly sharp in that respect: as the temperature increases, the system undergoes a sequence of phase transitions where dyons of charge (m, n) condense when m/n approaches θ [13]. When θ is irrational, there is an infinite number of phase transitions as $T \to \infty$. Of course, for all practical purposes, T is bounded from above and only a finite number of them are within reach. Other examples where the rationality of θ plays an important role include the "new" decoupled six-dimensional gauge theories [14], and the non-commutative open string in 1+1 dimensions [15]. In these cases, the very definition of the model with irrational θ angle or string coupling, respectively, is already problematic, and it would be interesting to see if the methods used in this paper are applicable.

In the case of Yang-Mills theories on a non-commutative torus, the difference between rational and irrational (and even between different kinds of irrational number) can be told by going to high energy. Indeed, for θ irrational, it has been found that the cascade of Morita equivalent descriptions never terminates as the energy is increased [11]. In this paper, we shall give an explicit construction of this cascade of pictures, by relating it to the continued fraction approximation of θ.[2] For θ rational, this will establish the existence of a finite covering of the full energy range; for θ irrational we will use it to study the non-local UV behavior of the gauge theory. Depending on some properties of irrational numbers and the level of confidence we require, we will see that one may or may not maintain a local description at short distance. In particular, we will show that there exist values of θ for which no quasi-local description exists beyond a certain energy. This is very similar to the case of $\mathbb{Z}_N$ lattice gauge theories in 4 dimensions, where Coulomb phases occur when no dyon can condense, and may prevail all the way to infinite temperature.

[2]The relevance of continued fraction expansion to non-commutative theories has been independently noticed in [16], with different motivations.

At strong 't Hooft coupling, we will find that a similar behavior holds: for some irrational values of Neveu-Schwarz B-field, there exists ranges of the radial coordinate for which no T-dual picture can give a local (and convergent) supergravity description. In fact, we shall argue that for θ irrational, the radial dimension becomes multifractal near the boundary of AdS. This is an interesting infrared effect in gravity theories with irrational NS B-field, which deserves further study.

The organization of this paper is as follow. In section 2, We shall take a look back at the problem of Villain $\mathbb{Z}_N$ lattice gauge theories, which will allow us to introduce the relevant mathematics. The cascade of Morita phases for NCSYM at weak coupling will then be constructed in section 3. In section 4, we shall discuss the UV behavior in the large 't Hooft coupling regime, from the point of view of the AdS/CFT correspondence, and briefly discuss the situation at finite temperature.

2. Phases of the Villain model

As a warm up, we reconsider the phase structure of four-dimensional $\mathbb{Z}_N$ lattice gauge theories [12] in the presence of a θ angle, following the discussion in [13]. These models are expected to capture universal features of U(N) Yang-Mills in the continuum. For simplicity, we consider a $\mathbb{Z}_N$ lattice gauge theory in the Villain approximation. The way to include a θ deformation reducing to the $\theta F \wedge F$ coupling in the continuum limit was discussed in [13]. The result is that the spectrum contains (m, n) states with arbitrary magnetic charge m and electric charge n. Due to the θ angle deformation, the effective electric charge $e = n - \theta m$ receives a fractional contribution from the magnetic charge through the Witten effect [17].

The phase structure is controlled by the type of states that condense in the vacuum: the Higgs phase corresponds to a condensation of electric states with charge $(0, 1)$ and hence confinement of magnetic charges; the ordinary confining phase corresponds to condensation of monopoles with charge $(1, 0)$. In addition, one may have oblique confinement whereby dyonic states of charge (m, n) condense [18]. If no state condenses, the gauge fields remain massless and we have a Coulomb phase. At weak coupling, or small temperature, there is always a Higgs phase. The structure at higher temperature depends however very sensitively on the value of θ. Indeed, a state with charge (m, n) condenses if its free energy is less than that of the vacuum. The free energy was estimated in [13], with the result that the condensation occurs if

$$(n - \theta m)^2\, T + \frac{m^2}{T} < \frac{C}{N}, \quad T = \frac{Ng^2}{2\pi}. \tag{2.1}$$

Here C is a numerical constant which can in principle be computed. The l.h.s. of this equation describes an ellipse in the $(m, e = n - m\theta)$ plane whose aspect ratio depends on the temperature T. The condensing states are the states of the lattice $\mathbb{Z} \times \mathbb{Z}$ contained in the ellipse and closest to the origin (see figure $1a$). For zero θ

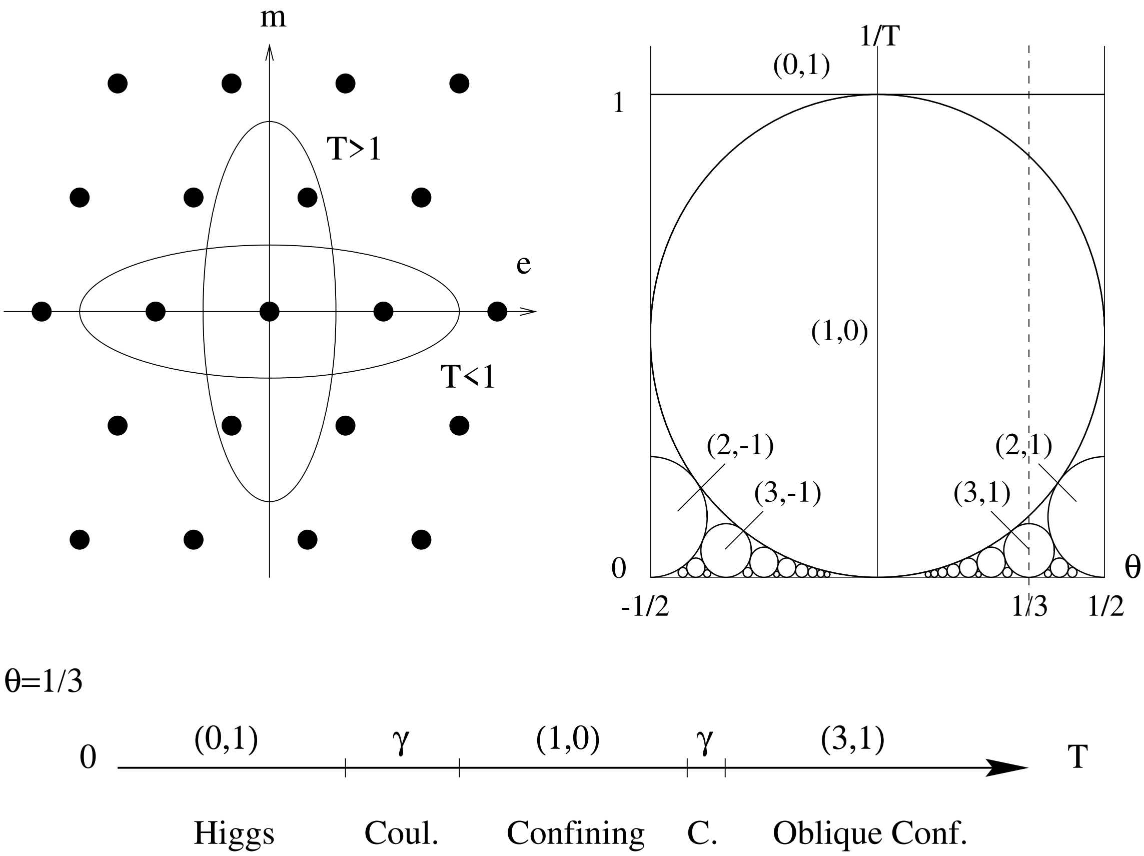

Figure 1: Left(a): Condensation ellipse in the (m, e) plane. The state contained in the ellipse and closest to the origin may condense at temperature T. Right(b): Condensation disks in the $(\theta, 1/T)$ plane for $M = 2$. The disks denote the domain where a (m, n) dyon may condense. Bottom(c): phase structure for $\theta = 1/3$.

angle, the electric state $(1, 0)$ clearly controls the low temperature behavior, while the magnetic state $(0, 1)$ controls the high temperature regime. The two phases may be connected for intermediate temperatures by a Coulomb phase if C/N is small enough so that the ellipse does not contain any non-zero lattice point.

The situation for non zero but rational $\theta = p/q$ is already quite richer. At very high temperature, the ellipse will select the dyonic states with charge $(m, n) = (q, p)$, which have zero effective electric charge $e = n - m\theta$: the high temperature phase will therefore be an oblique confining phase, in which the (q, p) dyon condenses. This will not occur however before $T > Nq^2/C$. Below this temperature, there will be other dyonic states with small effective magnetic charge $e = n_k - m_k\theta$, which will be able to condense at an intermediate temperature. If θ is irrational, this will in fact be the case at arbitrary high temperature: dyons with charge ration n_k/m_k arbitrarily close to θ will condense successively, and we will encounter an infinite number of phase transitions.

In more detail, there will exist a temperature T at which the state (m, n) will be able to condense if the minimum value of the l.h.s. in (2.1), attained at $T = 1/|\theta - n/m|$, is lower than the r.h.s, hence

$$|m(n - \theta m)| < \frac{1}{M}, \quad M = \frac{2N}{C}, \tag{2.2}$$

or equivalently

$$\left|\theta - \frac{n}{m}\right| < \frac{1}{Mm^2}. \tag{2.3}$$

If this inequality holds, the condensation will occur in a temperature interval

$$\frac{Mm^2}{2} < T < \frac{2}{M(n - m\theta)^2}, \tag{2.4}$$

where we assumed $|\theta - n/m| \ll 1$ for simplicity.

Evidently, the ability to approach θ by a rational number will determine the phase structure of the model. Fortunately, the mathematical topic of Diophantine approximation tells us a lot about the solutions of (2.3) (see for instance [19]). If $\theta = n/m$ is rational, it is known that there are only a finite number of fractions n_k/m_k approching θ to order $1/(Mm_k^2)$, irrespective of the constant M in (2.3). Hence there will be a finite number of phase transitions as T increases from 0 to ∞. In contrast, it is known that there is an infinite number of fractions n/m approaching any irrational θ to order $1/(Mm^2)$ with $M = 1$ at least. The case $M = 1$ is a simple application of Dirichlet's drawer principle, which guarantees that for any integer d there exists a fraction n/m with $m \leq d$ such that $|\theta - n/m| < 1/(md)$.

There is in fact an algorithm allowing to obtain an infinite number of solutions to (2.3), namely the continued fraction decomposition. Recall that any irrational number can be obtained as a limit of rational numbers θ_k, constructed out a series of positive integers (c_k) through the continued fraction

$$\theta_k = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots\ c_{k-1} + \cfrac{1}{c_k}}}}. \tag{2.5}$$

The series (c_k) specifies the real number θ uniquely. It truncates when θ is rational, and becomes periodic if θ is a quadratic irrational (i.e. a root of a second order polynomial with integer coefficients). The series $\theta_k = n_k/m_k$ obtained by truncating the fraction (2.5) at c_k converges monotonically to θ from above for k even, and from below for k odd. The integers n_k and m_k are strictly increasing and can be computed using the recursion relations

$$\begin{aligned} n_k &= c_k n_{k-1} + n_{k-2}, \quad n_0 = c_0, \; n_1 = c_0 c_1 + 1, \\ m_k &= c_k m_{k-1} + m_{k-2}, \quad m_0 = 1,, \; m_1 = c_1. \end{aligned} \tag{2.6}$$

Note that the recursion can also be initiated by $(n_{-1}, m_{-1}) = (1, 0)$. These equations preserve the relation

$$n_{k-1}m_k - n_k m_{k-1} = (-)^k \tag{2.7}$$

so that the integers (n_k, m_k) remain coprime. The fractions n_k/m_k are called the principal convergents of θ, while the fractions $n_k^{(p)}/m_k^{(p)} = (n_{k-2}+pn_{k-1})/(m_{k-2}+pm_{k-1})$ (with $p = 1 \cdots c_k - 1$) are called intermediate convergents and lie monotonically between n_{k-2}/m_{k-2} and n_k/m_k. The series n_k/m_k can be shown to realize the condition (2.3) for $M = 1$, i.e.

$$\left|\theta - \frac{n_k}{m_k}\right| \leq \frac{1}{m_k^2}\,. \tag{2.8}$$

In fact it can be shown that for any pair of consecutive convergents n_k/m_k and n_{k+1}/m_{k+1}, at least one of the two satisfies (2.3) for the stronger value $M = \sqrt{2}$; even better, for any triplet of consecutive convergents, at least one of the three satisfies (2.3) for the stronger value $M = \sqrt{5}$. This value cannot be improved, as one can show by taking θ to be the Golden ratio $(1 + \sqrt{5})/2$ (for which all $c_k = 1$). If θ is rational, the decomposition (2.5) terminates, and the n_k/m_k give the finite family of approximations to $\theta = n/m$ mentioned in the previous paragraph. Using $1/(m_k\theta - n_k)^2 \sim n_{k+1}^2/(m_k n_{k+1} - n_k m_{k+1})^2$ and (2.7), we see that the interval in (2.4) where the (n_k, m_k) state condenses is immediately followed by the one where (n_{k+1}, m_{k+1}) condenses, so that the entire temperature range is covered, by an infinite number of phases in the irrational case.

Whereas we have obtained solutions to (2.3) with M up to $\sqrt{5}$, the value of the constant C is not yet known, and may require finer rational approximations to θ. The ability to find solutions of (2.3) with $M > \sqrt{5}$ depends crucially on the irrationality of θ. To be more precise, let $M(\theta)$ define the supremum of all $M \in \mathbb{R}^+$ such that there exist an infinity of fractions n/m approching θ closer than $1/(Mm^2)$. Note that $M(\theta)$ is invariant under fractional linear transformations $\theta \to (a\theta + b)/(c\theta + d)$. It can in fact be shown that $M(\theta) = \sqrt{5}$ if and only if θ is equivalent to the Golden Ratio $\theta_1 = (1 + \sqrt{5})/2$ up to a $\mathrm{Sl}(2, \mathbb{Z})$ transformation. If not, then $M(\theta) \geq 2\sqrt{2}$, with equality iff θ is equivalent to $\theta_2 = 1 + \sqrt{2}$. More generally, there exists an infinite series (θ_k) with increasing $M(\theta_n) \to 3$, such that any θ with $M(\theta) < 3$ is equivalent to one of the θ_n's. These numbers θ_n form the Markoff chain, and are quadratic algebraic numbers, roots of second order polynomials which can be obtained in a recursive way [19]. Finally, most irrational numbers satisfy $M(\theta) \geq 3$, with infinitely many such that $M(\theta) = 3$. This state of affairs is represented on figure 2. Coming back to the physical problem at hand, if $C/(2N) > 1/M(\theta)$, there will be an infinity of states condensing as the temperature is increased from 0 to ∞. If however $C/(2N) < 1/M(\theta)$, as always occurs at large enough N, there will only be a finite number of phase transitions. This means that the system will reach a Coulomb phase at high temperature, after passing a finite number of phase transitions. This

JHEP10(2000)011

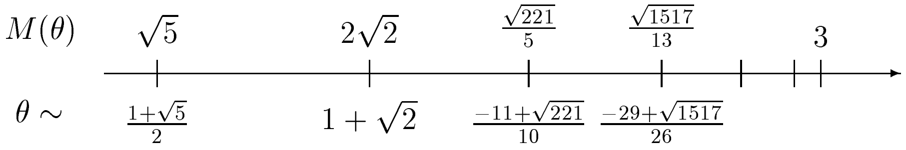

Figure 2: Maximum M achievable in (2.3) while preserving an infinity of solutions. $M(\theta) \geq 3$ except for a discrete series of quadratic irrationals θ_n.

is a rather unusual case where, even at infinite temperature, the symmetry is not restored. The system behaves at high temperature as if the gauge group was non compact, and thus did not possess topological excitations. The reader should however keep in mind that this statement holds within the Villain approximation.

Finally, we would like to mention an alternate way of understanding the phase structure of the abelian model. We first note following [13] that the free-energy in (2.1) is invariant under electric-magnetic duality, acting by $\mathrm{Sl}(2,\mathbb{Z})$ fractional linear transformations on the complexified coupling $\tau = \theta + i/T$,

$$\tau \longrightarrow \frac{c+d\tau}{a+b\tau}\,, \qquad \begin{pmatrix} m \\ n \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}, \qquad ad - bc = 1\,. \tag{2.9}$$

The temperature and θ angle can therefore always be mapped back into the fundamental domain of the upper half plane, $|\theta| < 1/2$, $|\tau| > 1$. For $\theta = 0$, we need only two patches $T > 1$ and $T < 1$ to cover the semi-infinite line $T > 0$, and the Higgs and confining phases mentioned above are simply related to each other by duality $T \to 1/T$. For a fixed non-zero rational θ, it takes finitely many patches to cover the line $\tau = \theta + i\mathbb{R}^+$, and hence we expect finitely many phase transitions. For an irrational θ, it takes an infinite number of patches, and hence we expect infinitely many phase transitions as $T \to \infty$. Each (m,n) dyon defines a disk on the upper half place, centered at $\tau = n/m + i/(Mm^2)$ and radius $1/(Mm^2)$ (hence tangent to the real axis) which delimits the region of temperature and θ angle where it might condense (see figure 1, right). The space between the disks correspond to Coulomb phases. For the critical value $M = 2$, all disks are tangent to each other. Following a line $\tau = \theta + i\mathbb{R}^+$ from infinity, one encounters an infinite number of phase transitions, except when θ is rational. For $M > 2$, the disks do not touch each other, and one may find a line such which does not encounter any disk as $T \to \infty$. For $M < 2$, the disks start overlap, in which case it is the state with the lowest free energy that condenses. For $M < \sqrt{3}$, the disks cover all the upper-half-plane, and there are no Coulomb phases any more.

Having used the Villain model as a setting to introduce the mathematical properties we need, we now come to the proper topic of this note, namely gauge theories on a non-commutative torus.

3. NCSYM on an irrational torus at weak coupling

Following [11], let us consider the $N = 4$ supersymmetric U(N) gauge theory on $\mathbb{R}^2 \times \mathbb{T}^2_\theta$ where $\mathbb{T}^2_\theta$ is a non-commutative torus with non-locality scale Δ and radius Σ. The dimensionless deformation parameter is $\theta = \Delta^2/\Sigma^2$. We also allow for a magnetic background m on $\mathbb{T}^2$. As we recalled in the introduction, the non-commutative gauge theory gives a quasi-local description at energies

$$\frac{1}{\Sigma} < E < \frac{\Sigma}{\Delta^2} = \frac{1}{\theta\Sigma}\,. \tag{3.1}$$

At either end of this interval, the wave function starts spreading over the whole torus $\mathbb{T}^2$, either in the direction of motion (at the lower end) or in the direction transverse to it (at the upper end). In the lower half $E < 1/\Delta$ of the interval (3.1), the effect of the non-commutativity is not noticeable, but this does not affect the statement of quasi-locality and we will not elaborate on this point here. These finite size effects may be avoided by looking for a Morita dual picture which brings the energy E within the new quasi-local window $1/\tilde{\Sigma} < E < 1/\tilde{\theta}\tilde{\Sigma}$. An Sl$(2,\mathbb{Z})$ Morita transformation acts as

$$\begin{gathered}\tilde{\theta} = \frac{c+d\theta}{a+b\theta}\,, \qquad \tilde{\Phi} = (a+b\theta)^2\Phi - b(a+b\theta)\,, \qquad \tilde{\Sigma} = (a+b\theta)\Sigma\,, \\ \tilde{g}^2_{\rm YM} = (a+b\theta)g^2_{\rm YM}\,, \qquad \begin{pmatrix}\tilde{m}\\ \tilde{N}\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} m\\ N\end{pmatrix},\end{gathered} \tag{3.2}$$

where Φ is the standard magnetic background [8, 10]. We thus have to look for integers a, b, c, d such $ad - bc = 1$ and

$$\frac{1}{|a+b\theta|\Sigma} < E < \frac{1}{|c+d\theta|\Sigma}\,. \tag{3.3}$$

If θ is rational, we may choose $\theta = -c/d$ such that the upper bound is infinite. The quasi-local description in the UV is therefore an ordinary commutative gauge theory, albeit with a magnetic flux. If θ is irrational, the upper bound can never be made infinite, and we have to keep finding dual descriptions as we increase the energy.

Given a fixed energy E, we may ask what is the optimum duality picture which allows to have a quasi-local description in the largest interval possible around E. In order to optimize the upper bound, we choose $|\theta + c/d| \ll 1$. Using $ad - bc = 1$, we can rewrite the inequality (3.3) as

$$d < E\Sigma < \frac{1}{|c+d\theta|}\,. \tag{3.4}$$

This is equivalent to

$$|\theta + c/d| < \frac{1}{d\,E\Sigma}\,, \quad d \le E\Sigma\,. \tag{3.5}$$

By the same Dirichlet's drawer principle mentioned in section 1, it is therefore clear that for any E there exists a fraction c/d such that the above condition is verified. Note that the two conditions in (3.5) imply

$$\left|\theta + \frac{c}{d}\right| < \frac{1}{d^2} \tag{3.6}$$

which does not make any reference to the energy any more. In particular, the continued fraction convergents of θ fulfill this condition but it is easy to see that they are not the only ones. The principal and intermediate convergents of θ however have the property of being *best approximations* to a given rational or irrational θ, in the sense that any closer fraction has to have a bigger denominator. By choosing c/d as a convergent of θ, we thus minimize the lower bound of (3.5) while maximizing the higher bound. Note that $a + b\theta$ tends to zero at the same time as $c + d\theta$ does, so that the Yang-Mills coupling $\tilde{g}_{\rm YM}$ keeps decreasing as we go to the UV. The 't Hooft coupling $\lambda = g^2_{\rm YM}(N - \theta m)$ however remains unchanged under Morita transformations.

Having justified the relevance of continued fraction approximations, we now proceed to show that the series of convergents n_k/m_k provides a complete covering of the full energy range. From the relation (2.7), we see that the linear fractional transformation

$$\theta \longrightarrow (-)^{n-1}\frac{m_k\theta - n_k}{m_{k-1}\theta - n_{k-1}} \tag{3.7}$$

is a bona fide $\mathrm{Sl}(2,\mathbb{Z})$ Morita transformation. Using the formulae (3.2), we find that the new range of validity of this dual picture

$$\frac{1}{|n_{k-1} - m_{k-1}\theta|\Sigma} < E < \frac{1}{|n_k - m_k\theta|\Sigma}\,. \tag{3.8}$$

Since $n_k - m_k\theta \to 0$ as k goes to infinity, we see that we have succeeded in covering the entire energy range with non-overlapping dual descriptions, for any irrational θ. In analogy with our discussion of the Villain model, one may want to introduce a confidence level in the criterion (3.1), and ask for which irrational values of θ the cascade of quasi-local descriptions can still be constructed. We will address this question in the next section, after discussing the strong coupling version of the criterion (3.1). The results will be the same, as far as the θ dependence is concerned, as those to be uncovered at strong coupling.

4. Strongly coupled NCSYM and multifractal gravity dual

We now turn to the strong 't Hooft coupling of NCSYM on a torus with irrational θ, which we can study using the AdS/CFT correspondence. We assume the 't Hooft coupling λ and $GCD(N, m)$ to be large. The metric, dilaton and B-field dual to

$N = 4$ SYM on a non-commutative torus have been obtained in [20, 21], and read

$$ds^2 = l_s^2 \left\{ \frac{U^2}{\sqrt{\lambda}}(-dt^2 + dx_1^2) + \frac{\sqrt{\lambda}U^2}{\lambda + U^4\Delta^4}(dx_2^2 + dx_3^2) + \frac{\sqrt{\lambda}}{U^2}dU^2 + \sqrt{\lambda}d\Omega_5^2 \right\},$$

$$e^\phi = \frac{\lambda}{4\pi N}\sqrt{\frac{\lambda}{\lambda + \Delta^4 U^4}}, \qquad B_{23} = -\frac{l_s^2 \Delta^2 U^4}{\lambda + \Delta^4 U^4}, \tag{4.1}$$

with periodicities $x_2 \sim x_2 + 2\pi\Sigma$ and $x_3 \sim x_3 + 2\pi\Sigma$. This gravity solution is appropriate for vanishing magnetic background $m = 0$, which we assume from now on. This is no loss of generality, since one may always choose $m = 0$ and $|\theta| < 1/2$ through a duality transformation [22].

The supergravity can be trusted when the volume of the torus is bigger than the string scale, hence

$$v = \frac{\sqrt{\lambda}U^2\Sigma^2}{\lambda + U^4\Delta^4} > \frac{M}{2}, \tag{4.2}$$

where the r.h.s. is the level of confidence we wish to impose on the supergravity description. If this condition is not satisfied, the Kaluza-Klein modes of the supergravity fields mix with the string winding states, and a supergravity description is no more sufficient. One may be tempted to identify M with the radius of convergence of the α' expansion, but the derivative expansion is presumably only asymptotic, so that this identification cannot be precise. We shall keep with the somewhat loose notion of "confidence level" in the following, although a better understanding of the constant M would certainly be desirable.

In terms of the dimensionless variable $u = U\Sigma/\lambda^{1/4}$, the volume of the torus in string units becomes

$$v = \frac{u^2}{(1 + \theta^2 u^4)} \tag{4.3}$$

which attains its maximum value $1/(2|\theta|)$ at $u = 1/\sqrt{|\theta|}$. The torus is therefore bigger than the string length for $|\theta| < 1/2$, in the range $u_- < u < u_+$ with $u_\pm^2 = (1 \pm \sqrt{1 - M^2\theta^2})/(M\theta^2)$. If $\theta \ll 1$, the supergravity solution can therefore be trusted in the range

$$\frac{\lambda^{1/4}}{\Sigma}\sqrt{\frac{M}{2}} < U < \frac{\lambda^{1/4}\Sigma}{\Delta^2}\sqrt{\frac{2}{M}} \tag{4.4}$$

with maximum reliability at $U = \lambda^{1/4}/\Delta$. Using the AdS/CFT dictionary and choosing $M = 2$, this translates into an energy range

$$\frac{1}{\lambda^{1/4}\Sigma} < E < \frac{\Sigma}{\lambda^{1/4}\Delta^2} \tag{4.5}$$

which agrees with the field theory criterion (3.1) up to a factor of $\lambda^{1/4}$, as observed in [11]. The requirement of quasi-locality at strong 't Hooft coupling (i.e. neglecting the non-local string winding states around the gravity background) therefore matches the requirement of quasi-locality at weak coupling, up to the usual large-N renormalisation by the factor of 't Hooft coupling. Note that the confidence level M could also have been introduced in the weak coupling criterion (3.1) by specifying the size of the energy window.

Outside this energy range, we may still find a reliable supergravity description by going to a T-dual picture (see [23] for a review). Applying an $\mathrm{Sl}(2,\mathbb{Z})$ T-duality transformation,

$$\rho \longrightarrow \frac{a\rho+b}{c\rho+d}\,, \qquad \rho=\frac{\Sigma^2}{l_s^2}B_{23}+iv\,, \qquad ve^{-2\phi}=\text{const.} \tag{4.6}$$

the volume of the dual torus in string units and the dilaton become

$$v=\frac{u^2}{d^2+(d\theta-c)^2u^4}\,, \qquad e^{\phi}=\frac{\lambda}{4\pi N}\frac{1}{\sqrt{d^2+(d\theta-c)^2u^4}}\,. \tag{4.7}$$

In particular, the string coupling remains smaller than the one in the original picture, at least by a factor $1/d$. Requiring again the volume to be bigger than a fixed constant $M/2$, we obtain a condition

$$\frac{d^2}{u^2}+(d\theta-c)^2u^2<\frac{2}{M} \tag{4.8}$$

isomorphic to the condition (2.1) appearing in the Villain model, under the identification

$$\tau=\theta+\frac{i}{T}\ (\text{Villain}) \longleftrightarrow \tau=\theta+\frac{i}{u^2}\ (\text{AdS})\,. \tag{4.9}$$

The constant-θ vertical line in the phase diagram of the Villain model (figure 1b) is therefore identified with the radial direction of AdS for a fixed value of the NS B-field. The disks now indicate what is the appropriate T-dual description (if any) for any point along the u axis. From (4.1), the metric along the radial direction is $(du/u)^2$, so that the phase structure is better represented in the $\theta+i\ln(u)$ plane, which corresponds to proper distance in target space (figure 3).

From this picture, we see that the interior of AdS ($u\to 0$, i.e. the infrared regime of the gauge theory) is universally described by the $(c,d)=(1,0)$ picture, where the volume of the torus diverges at $u\to 0$. The system is then describable by a purely D1-brane system (i.e. $N=0$), and undergoes a Gregory-Laflamme transition [24]: this is the standard phase transition from $3+1$ NCSYM to $1+1$ SYM [25]. In contrast, at the boundary of AdS ($u\to\infty$, i.e. to the UV of the field theory), the behavior depends crucially on the irrationality of θ. For any rational value of θ, the radial direction ends up into one of the shaded areas (i.e. one of the cusps at the bottom of figure 3), and remains there all the way towards the boundary of AdS. However, for θ irrational, the radial direction keeps switching from one shaded area to another. If $M>2$, the shaded areas are not connected, and there exists values

JHEP10(2000)011

Figure 3: Geometry of the radial dimension U (measured in proper units) with respect to the NS B-field. The shaded area is where a dual local supergravity description holds (we chose $M = 2.5$ for clarity of the picture). The area displayed is $1/8 < \theta < 1/2$ and $-7 < \ln(1/u^2) < -2$.

of θ for which there is no reliable description in some radial range. If $M < \sqrt{3}$, the complete τ plane is covered, and there is always a trustable T-dual description. Depending on the value of $M(\theta)$, the domain of radial positions for which a reliable description exists may be finite ($M(\theta) < M$) or infinite ($M(\theta) > M$). Note that the string coupling is not an issue, since e^{ϕ} decreases to zero as d increases to infinity. As in (2.2) and (2.3), the dual supergravity description has a non-empty range of radial positions in which it can be trusted if the minimum of the volume v is larger than $M/2$, i.e.

$$\frac{1}{2|d(d\theta - c)|} > \frac{M}{2}. \tag{4.10}$$

The range of validity is then

$$|d|\sqrt{\frac{M}{2}} < \lambda^{1/4} U \Sigma < \frac{1}{|d\theta - c|}\sqrt{\frac{2}{M}}. \tag{4.11}$$

The latter condition can be rewritten as $|\theta - c/d| < 1/(Md^2)$, which admits a finite ($M(\theta) < M$) or infinite ($M(\theta) > M$) number of solutions. The condition (4.11) for $M = 2$ agrees with the field theory condition (3.1) up to the coupling factor $\lambda^{1/4}$, upon noting that $d \sim 1/(a + b\theta)$ if $|\theta - c/d| \ll 1$.

Quite strikingly, the picture reveals the fractal nature of the radial dimension for irrational θ at the boundary of AdS. For θ a quadratic irrational, the continued fraction decomposition is periodic, and so is the phase structure on the radial direction (in logarithmic units): the radial direction has therefore a definite fractal dimension, related to the periodicity of the expansion. For irrational numbers that

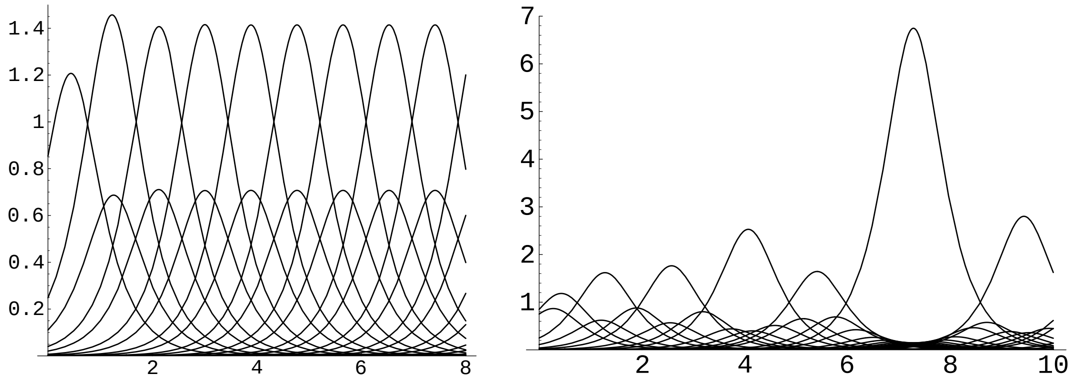

Figure 4: Volume of $\mathbb{T}^2$ in string units versus radial distance in logarithmic units for dual pictures given by the convergents and intermediate convergents of $\theta = \sqrt{2} - 1$ (left) and $\theta = \gamma_E \sim .577216$ (right).

are not quadractic however, the ratio of the energies at the branching points is quite erratic, and the radial dimension turns out to be multifractal (see [26] for a review). These two cases are illustrated on figure 4. If $M > M(\theta)$, the radial direction is completely stringy towards the boundary of AdS, and the dimension of the radial direction covered by duality pictures is therefore 0. It would be very interesting to compute the Hausdorff dimension and the spectrum of fractal exponents for general θ in terms of M and $M(\theta)$. As a word of caution, we emphasize that the fractality of the radial dimension only arises towards the boundary of AdS, i.e. as an infrared effect in the gravity theory. Whether non-commutativity can also be relevant for the description of the ultraviolet behavior of quantum gravity remains to be seen.

We conclude with three further comments on the phase structure of NCSYM. In the situation we have considered, there is no phase transition between different patches, but only a cross-over in the high energy behavior. One may ask whether this picture changes when we put the gauge theory on a non-commutative torus at finite temperature. The gravity dual was obtained in [21], and differs from (4.1) in the g_{tt} and g_{rr} components only, which are affected by a factor $(1 - U_0^4/U^4)$. The horizon at $U = U_0$ effectively cuts off the higher part of the diagram in figure 1b or figure 3, but leaves the ultraviolet behavior unchanged. The thermodynamic quantities are invariant under T-duality, so that we do not expect any phase transitions. Second, we may ask whether a Gregory-Laflamme transition can occur in the interior of AdS. This would in principle be achievable by tuning the ratio N/m close to θ. However, one may show that such a patch with only D1-branes ($N = 0$) always extends to $u \to 0$ [22]. The Gregory-Laflamme transition can therefore only occur in the infrared of the gauge theory. Finally, we should emphasize that the issue of quasi-locality concerns only the effective degrees of freedom at a given energy scale: local operators have smooth correlation functions at all scales, and may in fact become non local in the dual pictures that we introduced at higher energy.

Acknowledgments

The authors are grateful to J.L.F. Barbón and N. Seiberg for valuable discussions and suggestions. B.P. is indebted to N. Itzhaki for several discussions in Nov. 1999 on the UV behaviour of non-commutative gauge theories. E.R. is grateful to S. Shenker and S. Shenker for educating him on number theory in 1981, and pointing out its relevance to the Villain model, respectively. Hospitality and support of the Racah Institute of Physics in Jerusalem, the CERN Theory Division and the Tokyo Institute of Technology, Meguro, Tokyo 152-8551, extended to B.P. and E.R., respectively during part of this work are gratefully acknowledged. This research is supported in part by the European RTN network HPRN-CT-2000-00122, the David and Lucile Packard foundation, the BSF-American-Israeli Bi-National Science Foundation, the IRF Centers of Excellence Program and the Japan Society for the promotion of sciences.

References

[1] A. Gonzalez-Arroyo and C.P. K. Altes, *Reduced model for large-N continuum field theories*, *Phys. Lett.* **B 131** (1983) 396;
A. Connes and M. Rieffel, *Yang-Mills for noncommutative two-tori*, Operator algebras and Mathematical Physics, Iova City, 1985, pp. 237–266, *Contemp. Math. Oper. Algebra Math. Phys.* **62** AMS 1987.

[2] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[3] T. Filk, *Divergencies in a field theory on quantum space*, *Phys. Lett.* **B 376** (1996) 53;
J.C. Varilly and J.M. Gracia-Bondia, *On the ultraviolet behaviour of quantum fields over noncommutative manifolds*, *Int. J. Mod. Phys.* **A 14** (1999) 1305 [hep-th/9804001];
M.M. Sheikh-Jabbari, *Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus*, *J. High Energy Phys.* **06** (1999) 015 [hep-th/9903107];
C.P. Martin and D. Sanchez-Ruiz, *The one-loop UV divergent structure of* U(1) *Yang-Mills theory on noncommutative* R^4, *Phys. Rev. Lett.* **83** (1999) 476 [hep-th/9903077];
T. Krajewski and R. Wulkenhaar, *Perturbative quantum gauge fields on the noncommutative torus*, *Int. J. Mod. Phys.* **A 15** (2000) 1011 [hep-th/9903187].

[4] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[5] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[6] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[7] M.M. Sheikh-Jabbari, *Open strings in a B-field background as electric dipoles*, *Phys. Lett.* **B 455** (1999) 129 [hep-th/9901080];
D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, *Phys. Rev.* **D 62** (2000) 066004 [hep-th/9908056].

[8] A. Schwarz, *Morita equivalence and duality*, *Nucl. Phys.* **B 534** (1998) 720 [hep-th/9805034];
M. Rieffel and A. Schwarz, *Morita equivalence of multidimensional noncommutative tori*, q-alg/9803057.

[9] P.-M. Ho, *Twisted bundle on quantum torus and BPS states in matrix theory*, *Phys. Lett.* **B 434** (1998) 41 [hep-th/9803166];
D. Bigatti, *Non commutative geometry and super Yang-Mills theory*, *Phys. Lett.* **B 451** (1999) 324 [hep-th/9804120];
G. Landi, F. Lizzi and R.J. Szabo, *String geometry and the noncommutative torus*, *Comm. Math. Phys.* **206** (1999) 603 [hep-th/9806099];
B. Morariu and B. Zumino, *Super Yang-Mills on the noncommutative torus*, hep-th/9807198;
C. Hofman, E. Verlinde and G. Zwart, *U-duality invariance of the four-dimensional Born-Infeld theory*, *J. High Energy Phys.* **10** (1998) 020 [hep-th/9808128];
C. Hofman and E. Verlinde, *U-duality of Born-Infeld on the noncommutative two-torus*, *J. High Energy Phys.* **12** (1998) 010 [hep-th/9810116]; *Gauge bundles and Born-Infeld on the noncommutative torus*, *Nucl. Phys.* **B 547** (1999) 157 [hep-th/9810219].

[10] A. Konechny and A. Schwarz, *BPS states on noncommutative tori and duality*, *Nucl. Phys.* **B 550** (1999) 561 [hep-th/9811159];
A. Konechny and A. Schwarz, *Supersymmetry algebra and BPS states of super Yang-Mills theories on noncommutative tori*, *Phys. Lett.* **B 453** (1999) 23 [hep-th/9901077];
A. Konechny and A. Schwarz, *1/4-BPS states on noncommutative tori*, *J. High Energy Phys.* **09** (1999) 030 [hep-th/9907008];
B. Pioline and A. Schwarz, *Morita equivalence and T-duality (or B versus θ)*, *J. High Energy Phys.* **08** (1999) 021 [hep-th/9908019].

[11] A. Hashimoto and N. Itzhaki, *On the hierarchy between non-commutative and ordinary supersymmetric Yang-Mills*, *J. High Energy Phys.* **12** (1999) 007 [hep-th/9911057].

[12] S. Elitzur, R.B. Pearson and J. Shigemitsu, *The phase structure of discrete abelian spin and gauge systems*, *Phys. Rev.* **D 19** (1979) 3698;
D. Horn, M. Weinstein and S. Yankielowicz, *Hamiltonian approach to $\mathbb{Z}_n$ lattice gauge theories*, *Phys. Rev.* **D 19** (1979) 3715;
A. Ukawa, P. Windey and A.H. Guth, *Dual variables for lattice gauge theories and the phase structure of $\mathbb{Z}_N$ systems*, *Phys. Rev.* **D 21** (1980) 1013.

[13] J.L. Cardy and E. Rabinovici, *Phase structure of $\mathbb{Z}_p$ models in the presence of a θ parameter*, *Nucl. Phys.* **B 205** (1982) 1;
J.L. Cardy, *Duality and the theta parameter in abelian lattice models*, *Nucl. Phys.* **B 205** (1982) 17.

[14] E. Witten, *New 'gauge' theories in six dimensions*, *J. High Energy Phys.* **01** (1998) 001 [hep-th/9710065];
B. Kol, *On 6d 'gauge' theories with irrational θ angle*, *J. High Energy Phys.* **11** (1999) 017 [hep-th/9711017];
M. Alishahiha and Y. Oz, *Supergravity and 'new' six-dimensional gauge theories*, hep-th/0008172.

[15] R. Gopakumar, S. Minwalla, N. Seiberg and A. Strominger, *OM theory in diverse dimensions*, *J. High Energy Phys.* **08** (2000) 008 [hep-th/0006062];
I.R. Klebanov and J. Maldacena, *$1+1$ dimensional ncos and its $\mathrm{U}(N)$ gauge theory dual*, hep-th/0006085.

[16] G. Landi, F. Lizzi and R.J. Szabo, *From large-N matrices to the noncommutative torus*, hep-th/9912130.

[17] E. Witten, *Dyons of charge $e\,\theta/2\pi$*, *Phys. Lett.* **B 86** (1979) 283.

[18] G. 't Hooft, *Topology of the gauge condition and new confinement phases in nonabelian gauge theories*, *Nucl. Phys.* **B 190** (1981) 455.

[19] J. W. S. Cassels, *An introduction to Diophantine approximation*, Haffner Pub. Co., New York 1957, reprinted 1972;
S. Lang, *An introduction to Diophantine approximations*, Addison-Wesley Pub. Co. 1966.

[20] A. Hashimoto and N. Itzhaki, *Non-commutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166].

[21] J.M. Maldacena and J.G. Russo, *Large-N limit of non-commutative gauge theories*, *J. High Energy Phys.* **09** (1999) 025 [hep-th/9908134].

[22] J.L.F. Barbón, private communication.

[23] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rep.* **244** (1994) 77 [hep-th/9401139].

[24] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837 [hep-th/9301052];
The instability of charged black strings and p-branes, *Nucl. Phys.* **B 428** (1994) 399 [hep-th/9404071].

[25] J.L.F. Barbón, I.I. Kogan and E. Rabinovici, *On stringy thresholds in SYM/AdS thermodynamics*, *Nucl. Phys.* **B 544** (1999) 104 [hep-th/9809033];

S.A. Abel, J.L.F. Barbón, I.I. Kogan and E. Rabinovici, *String thermodynamics in D-brane backgrounds*, *J. High Energy Phys.* **04** (1999) 015 [hep-th/9902058];
Some thermodynamical aspects of string theory, hep-th/9911004;
E. Martinec and V. Sahakian, *Black holes and the SYM phase diagram, 2*, *Phys. Rev.* **D 59** (1999) 124005 [hep-th/9810224].

[26] B. Mandelbrot, *Multifractals and 1/F noise: wild self-affinity in physics*, selected works (1963-1976), Springer Verlag, 1999.